Focusing of Maximum Vertex Degrees in Random Faulty Scaled Sector Graphs

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Abstract

In this paper we study the behavior of maximum out/in-degree of binomial/Poisson random scaled sector graphs in the presence of random vertex and edge faults. We prove the probability distribution of maximum degrees for random faulty scaled sector graphs with n vertices, where each vertex spans a sector of α radians, with radius $r_n \ll \sqrt{\ln n/n}$, become concentrated on two consecutive integers, under some natural assumptions of faulty probabilities.

Keywords: Random scaled sector graphs; Random geometric graphs; Stein-Chen; Extremes

1. Introduction

Wireless ad-hoc communication network of sensors is gaining increasing importance in telecommunication society [1, 10]. The random scaled sector graph model proposed in [5] is aiming to provide a tool for the analysis of routing and transmission of information in sensor networks communicating through optical devices or directional antennae. In this setting, a randomly scattered large amount of transmitters are located in some geographical area. A transmitter can orient its laser beam in any position of a prescribed scanning area of α radians, however, the transmitter can receive light from any point, within distance r, that is "looking" to it. Moreover, some hostile unexpected environments should be taken into account. Every transmitter (vertex) may have a fail probability understood as the sensor becomes inoperative resulting from mechanical damages or power drain. Every connection (out-edge) may also have a fail probability understood as the failure of communication because of bad weather or terrain obstacles. Whence a random faulty scaled sector graph (see Section 2) is dealt with in this paper, which incorporating these features and serves our object well.

In this paper we investigate extreme degrees of the above mentioned random digraph model. A focusing result (Theorem 1) turns out that the maximum out-/in-degrees of random faulty scaled sector graph are almost determined under some natural assumptions of fault probabilities (see below for details). A similar focusing phenomenon has been discovered in classical Erdös-Rényi random graph theory, see e.g.[3](chap.3). Theorem 1 adds to the asymptotic bound of maximum degree of [5] in the thermodynamic and subconnective regimes and by including Poisson point process. It extends the maximum degree focusing result in [13](chap.6) by considering digraphs and including faulty probabilities. It also in passing gives an answer to the open problems suggested in [6].

Poisson approximation by Stein-Chen method is used here as in [14], where Penrose incorporated the geometric clique number with scan statistic via a "clustering rule" giving a concentration result. The idea behind can be traced back to [12], for instance. We mention that if the clustering rule h is properly chosen, the geometric maximum degree

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is also contained in that framework. In fact, let $h(\mathcal{X}) = 0$ if $\mathcal{X}$ is not contained in some ball $B(x, r)$ and otherwise let $h(\mathcal{X})$ be equal to the maximum degree+1 of the geometric subgraph induced by $\mathcal{X}$. This h can be seen suffice the requirements in [14].

Compared with the more recent random scaled sector graphs, random geometric graph models are widely-studied (see [13] and references therein), in which connections are isotropic and thus undirected. some properties such as connectivity and layout problems of geometric graphs including faulty probability have been addressed, see e.g.[6, 8, 15, 17]. Algorithmic aspects are also dealt with in various contexts, see e.g.[11, 16]. From a percolation analysis point of view, some concerned results coherent with ours can be found in [7].

2. Statement of main results

Given a sequence $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$ of independently and uniformly distributed (*i.u.d.*) random points in $[0,1]^2$ with common density function $f = 1_{[0,1]^2}$. We equip $\mathbb{R}^2$ with Euclidean norm and let θ be the area of unit disk. Notice that $\theta = \pi$ in this context, and we may sometimes retain the sign θ for readability. Let $\mathcal{Y}_n = \{Y_1, Y_2, \cdots, Y_n\}$ be *i.u.d.* random variables taking values in $[0, 2\pi)$ and $\alpha \in (0, 2\pi]$ be fixed. Associate every point $X_i \in \mathcal{X}_n$ a sector, which is centered at X_i, with radius r_n, central angle α and elevation Y_i with respect to the horizontal direction anticlockwise. This sector is denoted as $S(X_i, Y_i, r_n)$. We denote by $G_\alpha(\mathcal{X}_n, \mathcal{Y}_n, r_n)$ the digraph with vertex set $\mathcal{X}_n$, and with arc (X_i, X_j), $i \neq j$, presents if and only if $X_j \in S(X_i, Y_i, r_n)$. The usual Poisson version $G(\mathcal{P}_n, \mathcal{Y}_{N_n}, r_n)$ is defined similarly, where $\mathcal{P}_n = \{X_1, X_2, \cdots, X_{N_n}\}$, $N_n \sim Poi(n)$. Now for every X_i, we assign a failure probability v_n, independent of other node and the point process. Given the presence of X_i, we associate every out-edge with a failure probability q_n independently. This random faulty scaled sector graph is thus denoted by $G_\alpha(\mathcal{X}_n, \mathcal{Y}_n, v_n, q_n, r_n)$. Likewise, we can define the Poisson version $G_\alpha(\mathcal{P}_n, \mathcal{Y}_{N_n}, v_n, q_n, r_n)$. Set Δ_n^{out}, Δ_n^{in} be the maximum out-/in-degree of $G_\alpha(\mathcal{X}_n, \mathcal{Y}_n, v_n, q_n, r_n)$ respectively, and $\Delta_n'^{out}$, $\Delta_n'^{in}$ be the maximum out-/in-degree of $G_\alpha(\mathcal{P}_n, \mathcal{Y}_{N_n}, v_n, q_n, r_n)$ respectively. Here comes the main result:

Theorem 1. *Suppose $v_n \to v \in [0,1)$ and $q_n \to q \in [0,1)$, as $n \to \infty$. Suppose $\mu_n := \frac{\alpha}{2} n r_n^2 (1 - v_n)(1 - q_n)$, and that $\inf_{n>0} \mu_n > 0$, and that $\mu_n^{1+\varepsilon} = o(\ln n)$ for some $\varepsilon > 0$. Then there exists a sequence $\{k_n\}_{n \geq 1}$, set $\xi_n = P(Poi(\mu_n) \geq k_n)$, such that we have*

$$P(\Delta_n'^{out} = k_n - 1) - e^{-n(1-v_n)\xi_n} \to 0,$$
$$P(\Delta_n'^{out} = k_n) - e^{-n(1-v_n)\xi_n} \to 1,$$

as $n \to \infty$. The same thing holds for $\Delta_n'^{in}$, Δ_n^{out} and Δ_n^{in}.

3. Proofs

To prove the asymptotic focusing phenomenon, we first give a general non-asymptotic Poisson approximation lemma, which may be useful in some other cases. Let $W_{j,n}^{out}(r)$, $W_{j,n}^{in}(r)$ be the number of vertices of out-/in-degree j in $G_\alpha(\mathcal{X}_n, \mathcal{Y}_n, v, q, r)$ respectively. Let $W_{j,\lambda}'^{out}(r)$, $W_{j,\lambda}'^{in}(r)$ be the number of vertices of out-/in-degree j in $G_\alpha(\mathcal{P}_\lambda, \mathcal{Y}_{N_\lambda}, v, q, r)$ respectively. For $A \subseteq \mathbb{N} \cup \{0\}$, set $W_{A,\lambda}'^{out}(r) := \sum_{j \in A} W_{j,\lambda}'^{out}(r)$ and $W_{A,\lambda}'^{in}(r) := \sum_{j \in A} W_{j,\lambda}'^{in}(r)$. The total variation distance between the laws of integer valued random variables X, Y is defined by

$$d_{TV}(X, Y) = \sup_{A \subseteq \mathbb{Z}} \{|P(X \in A) - P(Y \in A)|\}.$$

Let c, c' be various positive constants, the values may change from line to line.

Lemma 1. *Suppose a density function g (not necessarily uniform) is continuous a.e.. Let $0 \le v, q < 1$, $r, \lambda > 0$ and $A \subseteq \mathbb{N} \cup \{0\}$. Then,*

$$d_{TV}(W'^{out}_{A,\lambda}, Poi(EW'^{out}_{A,\lambda})) \le \min\Big(1, \frac{1}{EW'^{out}_{A,\lambda}}\Big)(I_1^{out} + I_2^{out})$$

where,

$$I_1^{out} := \frac{(1-v)^2\lambda^2}{4\pi^2} \int_{\mathbb{R}^2} \int_0^{2\pi} P[\mathcal{P}_{\lambda(1-q)(1-v)}(S(x_1, y_1, r) \in A)]\mathrm{d}y_1$$
$$\int_{B(x_1,3r)} \int_0^{2\pi} P[\mathcal{P}_{\lambda(1-q)(1-v)}(S(x_2, y_2, r) \in A)]\mathrm{d}y_2 g(x_2)\mathrm{d}x_2 g(x_1)\mathrm{d}x_1,$$

$$I_2^{out} := \frac{(1-v)^2\lambda^2}{4\pi^2}$$
$$\cdot \int_{\mathbb{R}^2} \int_{B(x_1,3r)} \int_0^{2\pi} \int_0^{2\pi} P\big[\{\mathcal{P}_{\lambda(1-q)(1-v)}(S(x_1, y_1, r)) + 1_{[x_2 \in S(x_1,y_1,r)] \cap [(x_1,x_2) \ not \ fails]} \in A\}$$
$$\cap \{\mathcal{P}_{\lambda(1-q)(1-v)}(S(x_2, y_2, r)) + 1_{[x_1 \in S(x_2,y_2,r)] \cap [(x_2,x_1) \ not \ fails]} \in A\}\big]$$
$$\mathrm{d}y_1 \mathrm{d}y_2 g(x_2)\mathrm{d}x_2 g(x_1)\mathrm{d}x_1$$

Likewise,

$$d_{TV}(W'^{in}_{A,\lambda}, Poi(EW'^{in}_{A,\lambda})) \le \min\Big(1, \frac{1}{EW'^{in}_{A,\lambda}}\Big)(I_1^{in} + I_2^{in})$$

where,

$$I_1^{in} := (1-v)^2\lambda^2 \int_{\mathbb{R}^2} P[\tilde{\mathcal{P}}_{\lambda(1-q)(1-v)}(B(x_1, r)) \in A]$$
$$\int_{B(x_1,3r)} P[\tilde{\mathcal{P}}_{\lambda(1-q)(1-v)}(B(x_2, r)) \in A] g(x_2)\mathrm{d}x_2 g(x_1)\mathrm{d}x_1,$$

$$I_2^{in} := (1-v)^2\lambda^2$$
$$\cdot \int_{\mathbb{R}^2} \int_{B(x_1,3r)} P\big[\{\tilde{\mathcal{P}}_{\lambda(1-q)(1-v)}(B(x_1, r)) + 1_{[x_1 \in S(x_2,y_2,r)] \cap [(x_2,x_1) \ not \ fails]} \in A\}$$
$$\cap \{\tilde{\mathcal{P}}_{\lambda(1-q)(1-v)}(B(x_2, r)) + 1_{[x_2 \in S(x_1,y_1,r)] \cap [(x_1,x_2) \ not \ fails]} \in A\}\big] g(x_2)\mathrm{d}x_2 g(x_1)\mathrm{d}x_1$$

and denote by $\tilde{\mathcal{P}}_\lambda$ with intensity $(\lambda\alpha/2\pi)g$ the thinning of $\mathcal{P}_\lambda$, whose intensity is λg.

Proof. Given $m \in \mathbb{N}$, partition $\mathbb{R}^2$ into squares of side 2^{-m}, with the origin lies at a square corner. Label these squares as $D_{m,1}, D_{m,2}, \cdots$, and denote the center of $D_{m,i}$ as $a_{m,i}$. For each $x \in \mathbb{R}^2$ and for each m, i, define Y_x, $Y_{m,i}$ as independent copies of Y_1.

For out-degree, set

$$\xi_{m,i} := 1_{[\mathcal{P}_{\lambda(1-v)}(D_{m,i})=1] \cap [\mathcal{P}_{\lambda(1-q)(1-v)}(S(a_{m,i}, Y_{m,i}, r) \setminus D_{m,i}) \in A]}$$

Set $p_{m,i} := E\xi_{m,i}$, $p_{m,i,j} := E[\xi_{m,i}\xi_{m,j}]$. Define an adjacency relation $\sim_m$ on $\mathbb{N}$ by putting $i \sim_m j$ if and only if $0 < ||a_{m,i} - a_{m,j}|| \le 3r$, and define the corresponding adjacency

neighborhood $\mathcal{N}_{m,i} := \{j \in \mathbb{N} |\ ||a_{m,i} - a_{m,j}|| \le 3r\}$. Let $Q_n := [-n, n]^2$ and $\mathcal{I}_{m,n} := \{i \in \mathbb{N} | D_{m,i} \subseteq Q_n\}$. Set $\mathcal{N}_{m,n,i} := \mathcal{N}_{m,i} \cap \mathcal{I}_{m,n}$. Thus $(\mathcal{I}_{m,n}, \sim_m)$ is a dependency graph for random variables $\xi_{m,i}$, $i \in \mathcal{I}_{m,n}$.

Define $\tilde{W}^{out}_{m,n} := \sum_{i \in \mathcal{I}_{m,n}} \xi_{m,i}$, then we observe that $W'^{out}_{A,\lambda} = \lim_{n\to\infty} \lim_{m\to\infty} \tilde{W}^{out}_{m,n}$. By theorem 1 of [2],

$$d_{TV}(\tilde{W}^{out}_{m,n}, Poi(E\tilde{W}^{out}_{m,n})) \le \min\Big(1, \frac{1}{E\tilde{W}^{out}_{m,n}}\Big)(a_1(m,n) + a_2(m,n)) \tag{1}$$

where

$$a_1(m,n) := \sum_{i \in \mathcal{I}_{m,n}} \sum_{j \in \mathcal{N}_{m,n,i}} p_{m,i} p_{m,j}, \qquad a_2(m,n) := \sum_{i \in \mathcal{I}_{m,n}} \sum_{j \in \mathcal{N}_{m,n,i} \setminus \{i\}} p_{m,i,j}.$$

Define $w_m(x) := 2^{2m} p_{m,i} 1_{[x \in D_{m,i}]}$, wherefore $\int_{Q_n} w_m(x) \mathrm{d}x = \sum_{i \in \mathcal{I}_{m,n}} p_{m,i}$. If f is continuous at x, we have $\lim_{m\to\infty} w_m(x) = (1-v)\lambda g(x) P[\mathcal{P}_{\lambda(1-q)(1-v)}(S(x, Y_x, r)) \in A]$, by mean value theorem. Observe that $w_m(x) \le 2^{2m} E\mathcal{P}_{\lambda(1-v)}(D_{m,i}) \le \lambda g_{\max}$, so by the dominated convergence theorem, we obtain

$$\lim_{m\to\infty} E\tilde{W}^{out}_{m,n} = (1-v)\lambda \int_{Q_n} P[\mathcal{P}_{\lambda(1-q)(1-v)}(S(x, Y_x, r)) \in A] g(x) \mathrm{d}x$$

and by Fubini theorem and Palm theory for (marked) Poisson point process [13](Sect. 1.7), we have

$$\begin{aligned}
\lim_{n\to\infty} \lim_{m\to\infty} E\tilde{W}^{out}_{m,n} &= (1-v)\lambda \int_{\mathbb{R}^2} P[\mathcal{P}_{\lambda(1-q)(1-v)}(S(x, Y_x, r)) \in A] g(x) \mathrm{d}x \\
&= \frac{(1-v)\lambda}{2\pi} \int_{\mathbb{R}^2} \int_0^{2\pi} P[Poi\big(\int_{S(x,y,r)} \lambda(1-v)(1-q) g(z) \mathrm{d}z\big) \in A] g(x) \mathrm{d}y \mathrm{d}x \\
&= EW'^{out}_{A,\lambda}
\end{aligned}$$

For $x_1 \in D_{m,i}$, $x_2 \in D_{m,j}$, define $u_m(x_1, x_2) := 2^{4m} p_{m,i} p_{m,j} 1_{[j \in \mathcal{N}_{m,i}]}$ and $v_m(x_1, x_2) := 2^{4m} p_{m,i,j} 1_{[j \in \mathcal{N}_{m,i} \setminus \{i\}]}$. Therefore, we have $a_1(m,n) = \int_{Q_n} \int_{Q_n} u_m(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2$ and $a_2(m,n) = \int_{Q_n} \int_{Q_n} v_m(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2$. For different continuous points x_1, x_2 of g, if also $||x_1 - x_2|| \ne r$ and $||x_1 - x_2|| \ne 3r$, then

$$\begin{aligned}
\lim_{m\to\infty} u_m(x_1, x_2) = \frac{(1-v)^2\lambda^2}{4\pi^2} g(x_1) g(x_2) \int_0^{2\pi} P[\mathcal{P}_{\lambda(1-q)(1-v)}(S(x_1, y_1, r) \in A)] \mathrm{d}y_1 \\
\cdot \int_0^{2\pi} P[\mathcal{P}_{\lambda(1-q)(1-v)}(S(x_2, y_2, r) \in A)] \mathrm{d}y_2 \cdot 1_{[B(x_1, 3r)]}(x_2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\lim_{m\to\infty} v_m(x_1, x_2) = \frac{(1-v)^2\lambda^2}{4\pi^2} g(x_1) g(x_2) \\
&\quad \cdot \int_0^{2\pi} \int_0^{2\pi} P\big[\{\mathcal{P}_{\lambda(1-q)(1-v)}(S(x_1, y_1, r)) + 1_{[x_2 \in S(x_1, y_1, r)] \cap [(x_1, x_2)\ not\ fails]} \in A\} \\
&\cap \{\mathcal{P}_{\lambda(1-q)(1-v)}(S(x_2, y_2, r)) + 1_{[x_1 \in S(x_2, y_2, r)] \cap [(x_2, x_1)\ not\ fails]} \in A\}\big] \mathrm{d}y_1 \mathrm{d}y_2 \cdot 1_{[B(x_1, 3r)]}(x_2)
\end{aligned}$$

For $x_1 \in D_{m,i}$, $x_2 \in D_{m,j}$, we have

$$u_m(x_1, x_2) \le 2^{4m} E\mathcal{P}_{\lambda(1-v)}(D_{m,i}) E\mathcal{P}_{\lambda(1-v)}(D_{m,j}) \le g_{\max}^2$$

and

$$v_m(x_1, x_2) \le 2^{4m} E\mathcal{P}_{\lambda(1-v)}(D_{m,i}) E\mathcal{P}_{\lambda(1-v)}(D_{m,j}) 1_{[i \neq j]} \le g_{\max}^2.$$

Whence by the dominated convergence theorem we have $\lim_{n\to\infty} \lim_{m\to\infty} a_1(m,n) = I_1^{out}$ and $\lim_{n\to\infty} \lim_{m\to\infty} a_2(m,n) = I_2^{out}$. Hence, the out-degree case is proved by taking limit in both sides of (1).

For in-degree, set

$$\eta_{m,i} := 1_{[\mathcal{P}_{\lambda(1-v)}(D_{m,i})=1] \cap [\#\{x' \in \mathcal{P}_{\lambda(1-q)(1-v)} | D_{m,i} \subseteq S(x', Y_{x'}, r)\} \in A]}$$

Set $q_{m,i} := E\eta_{m,i}$, $q_{m,i,j} := E[\eta_{m,i}\eta_{m,j}]$. Define the dependency graph for random variables $\eta_{m,i}$, $i \in \mathcal{I}_{m,n}$ just as above.

Define $\tilde{W}_{m,n}^{in} := \sum_{i \in \mathcal{I}_{m,n}} \eta_{m,i}$, then we observe that $W_{A,\lambda}^{\prime in} = \lim_{n\to\infty} \lim_{m\to\infty} \tilde{W}_{m,n}^{in}$. By theorem 1 of [2],

$$d_{TV}(\tilde{W}_{m,n}^{in}, Poi(E\tilde{W}_{m,n}^{in})) \le \min\Big(1, \frac{1}{E\tilde{W}_{m,n}^{in}}\Big)(b_1(m,n) + b_2(m,n)) \tag{2}$$

where

$$b_1(m,n) := \sum_{i \in \mathcal{I}_{m,n}} \sum_{j \in \mathcal{N}_{m,n,i}} q_{m,i} q_{m,j}, \qquad b_2(m,n) := \sum_{i \in \mathcal{I}_{m,n}} \sum_{j \in \mathcal{N}_{m,n,i} \setminus \{i\}} q_{m,i,j}.$$

Reset $w_m(x) := 2^{2m} q_{m,i} 1_{[x \in D_{m,i}]}$, then $\int_{Q_n} w_m(x) \mathrm{d}x = \sum_{i \in \mathcal{I}_{m,n}} q_{m,i}$. If f is continuous at x, we have $\lim_{m\to\infty} w_m(x) = (1-v)\lambda g(x) P[\#\{x' \in \mathcal{P}_{\lambda(1-q)(1-v)} | x \in S(x', Y_{x'}, r)\} \in A]$, by mean value theorem for integrals. Observe that $w_m(x) \le 2^{2m} E\mathcal{P}_{\lambda(1-v)}(D_{m,i}) \le \lambda g_{\max}$, so by the dominated convergence theorem, we obtain

$$\lim_{m\to\infty} E\tilde{W}_{m,n}^{in} = (1-v)\lambda \int_{Q_n} P[\#\{x' \in \mathcal{P}_{\lambda(1-q)(1-v)} | x \in S(x', Y_{x'}, r)\} \in A] g(x) \mathrm{d}x$$

and

$$\begin{aligned}\lim_{n\to\infty} \lim_{m\to\infty} E\tilde{W}_{m,n}^{in} &= (1-v)\lambda \int_{\mathbb{R}^2} P[Poi\Big(\frac{\lambda\alpha(1-q)(1-v)}{2\pi} \int_{B(x,r)} g(z)\mathrm{d}z\Big) \in A] g(x) \mathrm{d}x \\ &= EW_{A,\lambda}^{\prime in}\end{aligned}$$

For $x_1 \in D_{m,i}$, $x_2 \in D_{m,j}$, reset $u_m(x_1, x_2) := 2^{4m} q_{m,i} q_{m,j} 1_{[j \in \mathcal{N}_{m,i}]}$ and $v_m(x_1, x_2) := 2^{4m} q_{m,i,j} 1_{[j \in \mathcal{N}_{m,i} \setminus \{i\}]}$. Therefore, we have $b_1(m,n) = \int_{Q_n} \int_{Q_n} u_m(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2$ and $b_2(m,n) = \int_{Q_n} \int_{Q_n} v_m(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2$. For different continuous points x_1, x_2 of g, if also $||x_1 - x_2|| \neq r$ and $||x_1 - x_2|| \neq 3r$, then

$$\begin{aligned}\lim_{m\to\infty} u_m(x_1, x_2) = (1-v)^2 \lambda^2 g(x_1) g(x_2) P[\tilde{\mathcal{P}}_{\lambda(1-q)(1-v)}(B(x_1, r) \in A)] \\ \cdot P[\tilde{\mathcal{P}}_{\lambda(1-q)(1-v)}(B(x_2, r) \in A)] 1_{[B(x_1, 3r)]}(x_2)\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{m\to\infty} v_m(x_1,x_2) = (1-v)^2\lambda^2 g(x_1)g(x_2)\\ \cdot P[\{\tilde{\mathcal{P}}_{\lambda(1-q)(1-v)}(B(x_1,r)) + 1_{[x_1\in S(x_2,Y_2,r)]\cap[(x_2,x_1)\ not\ fails]} \in A\}\\ \cap \{\tilde{\mathcal{P}}_{\lambda(1-q)(1-v)}(B(x_2,r)) + 1_{[x_2\in S(x_1,Y_1,r)]\cap[(x_1,x_2)\ not\ fails]} \in A\}] \cdot 1_{[B(x_1,3r)]}(x_2)\end{aligned}$$

By similar arguments in out-degree case, we have $\lim_{n\to\infty}\lim_{m\to\infty} b_1(m,n) = I_1^{in}$ and $\lim_{n\to\infty}\lim_{m\to\infty} b_2(m,n) = I_2^{in}$. Hence, we conclude the proof by taking limit in both sides of (2). □

For cleanness of the expressions, we will shift our battlefield from $[0,1]^2$ to $[-1/2,1/2]^2$. From now on, we take $f = 1_Q$, $Q := [-1/2,1/2]^2$ through this paper. Let $\mathcal{H}_\lambda$ be the homogeneous Poisson point process with intensity λ on $\mathbb{R}^2$ and $|\cdot|$ be Lebesgue measure.

Proposition 1. *Let $\mu_n := \frac{\alpha}{2}nr_n^2(1-v_n)(1-q_n)$, and suppose $\lim_{n\to\infty} r_n = 0$ and $\inf_{n>0}\mu_n > 0$. Suppose $\{j_n\}_{n\geq 1}$ is an $\mathbb{N}-$valued sequence such that for some $\varepsilon > 0$,*

$$\lim_{n\to\infty} j_n/\mu_n^{1+\varepsilon} = \infty \tag{3}$$

Set $\xi_n := P(Poi(\mu_n) \geq j_n)$. Then

$$\lim_{n\to\infty}[P(\Delta_n'^{out} < j_n) - e^{-n(1-v_n)\xi_n}] = 0$$

and

$$\lim_{n\to\infty}[P(\Delta_n'^{in} < j_n) - e^{-n(1-v_n)\xi_n}] = 0.$$

Proof. For out-degree, set

$$W_n'^{out} := \sum_{i=1}^{N_{n(1-v_n)}} 1_{[\mathcal{P}_{n(1-v_n)(1-q_n)}(S(X_i,Y_i,r_n))\geq j_n+1]\cap[X_i\in Q]}$$

Then by Palm theory for (marked) Poisson process,

$$EW_n'^{out} \sim (1-v_n)n\int_Q P[Poi(\frac{\theta\alpha}{2\pi}nr_n^2(1-v_n)(1-q_n)) \geq j_n]\mathrm{d}x = (1-v_n)n\xi_n,$$

as $n\to\infty$. Now take $A = \mathbb{Z}\cap[j_n,\infty)$, $\lambda = n$, $r = r_n$, $v = v_n$, $q = q_n$, $g = f$ in Lemma 1, we then obtain

$$\left|P(W_n'^{out} = 0) - e^{-EW_n'^{out}}\right| \leq \min\Big(1, \frac{2}{(1-v_n)n\xi_n}\Big)(I_{1,n}^{out} + I_{2,n}^{out}), \tag{4}$$

for large enough n. We have

$$\begin{aligned} I_{1,n}^{out} &= \frac{(1-v_n)^2n^2}{4\pi^2}\int_{\mathbb{R}^2}\int_0^{2\pi} P[\mathcal{P}_{n(1-q_n)(1-v_n)}(S(x_1,y_1,r_n)) \geq j_n]\mathrm{d}y_1\\ &\quad \int_{B(x_1,3r_n)}\int_0^{2\pi} P[\mathcal{P}_{n(1-q_n)(1-v_n)}(S(x_2,y_2,r_n)) \geq j_n]\mathrm{d}y_2 f(x_2)\mathrm{d}x_2 f(x_1)\mathrm{d}x_1\\ &\leq n^2\xi_n^2\theta(3r_n)^2\end{aligned}$$

Therefore, by (3), $((1-v_n)n\xi_n)^{-1}I_{1,n}^{out} \le c\mu_n\xi_n \to 0$. On the other hand, we have,

$$\begin{aligned}
I_{2,n}^{out} &= \frac{(1-v_n)^2n^2}{4\pi^2}\int_Q\int_{Q\cap B(x_1,3r_n)}\int_0^{2\pi}\int_0^{2\pi} P\big[\{\mathcal{P}_{n(1-q_n)(1-v_n)}(S(x_1,y_1,r_n)) \\
&\quad +1_{[x_2\in S(x_1,y_1,r_n)]\cap[(x_1,x_2)\ not\ fails]} \ge j_n\} \cap \{\mathcal{P}_{n(1-q_n)(1-v_n)}(S(x_2,y_2,r_n)) \\
&\quad +1_{[x_1\in S(x_2,y_2,r_n)]\cap[(x_2,x_1)\ not\ fails]} \ge j_n\}\big]\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}x_2\mathrm{d}x_1 \\
&\le \frac{(1-v_n)^2n^2}{4\pi^2}\int_Q\int_{Q\cap B(0,3r_n)}\int_0^{2\pi}\int_0^{2\pi} P\big[\{\mathcal{P}_{n(1-q_n)(1-v_n)}(S(0,y_1,r_n)) \ge j_n-1\} \\
&\quad \cap\{\mathcal{P}_{n(1-q_n)(1-v_n)}(S(x_2-x_1,y_2,r_n)) \ge j_n-1\}\big]\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}x_2\mathrm{d}x_1 \\
&\le \frac{(1-v_n)^2n^2}{4\pi^2}\int_{B(0,3)}\int_0^{2\pi}\int_0^{2\pi} h_n(z,y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z
\end{aligned}$$

where,

$$\begin{aligned}
h_n(z,y_1,y_2) := P\big[&\{\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(0,y_1,1)) \ge j_n-1\} \\
&\cap\{\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(z,y_2,1)) \ge j_n-1\}\big].
\end{aligned}$$

By (3), we choose $M\in\mathbb{N}$, such that $j_n^M/\mu_n^{M+1}\to\infty$, as $n\to\infty$. Then we have

$$P[\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(0,y_1,1)) \ge j_n+M] \le \xi_n\Big(\frac{\mu_n}{j_n}\Big)^M$$

and

$$P[\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(0,y_1,1)) \in \{j_n-1,j_n,\cdots,j_n+M-1\}] \le 2\xi_n\frac{j_n}{\mu_n}$$

since $P[\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(0,y_1,1)) = j_n-1] \le \xi_n(j_n/\mu_n)$ and $P[\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(0,y_1,1)) \ge j_n] \le \xi_n(j_n/\mu_n)$ when n is large enough.

Let $\eta_z := 2\pi|S(0,y_1,1)\backslash S(z,y_2,1)|/\theta\alpha$, then we see that the conditional distribution of $\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(z,y_2,1))$, given that $\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(0,y_1,1)) = j_n+M$ is the sum of two independent random variables j_n+M-U and V, where $U\sim Bin(j_n+M,\eta_z)$ and $V\sim Poi(\theta\alpha nr_n^2\eta_z(1-q_n)(1-v_n)/2\pi)$. Provided n is large enough so that $M+1 < j_n\eta_z/5$, if $U > 3j_n\eta_z/5$ and $V < j_n\eta_z/5$ then $j_n+M-U+V < j_n-1$. Now, note that $P[\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(z,y_2,1)) \ge j_n-1 | \mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(0,y_1,1)) = k]$ is an increasing function of k, and by Chernoff bounds, there exists a constant $\beta>0$, for any $z\in B(0,3)$ and n large enough, if $\eta_z > 5(M+1)/j_n$ then

$$\begin{aligned}
&\max_{j_n-1\le k\le j_n+M-1} P[\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(z,y_2,1)) \ge j_n-1 | \mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(0,y_1,1)) = k] \\
&\quad \le P[Bin(j_n+M,\eta_z) \le 3j_n\eta_z/5] + P[Poi(\theta\alpha nr_n^2\eta_z(1-q_n)(1-v_n)/2\pi) \ge j_n\eta_z/5] \\
&\quad \le 2e^{-\beta j_n\eta_z},
\end{aligned}$$

whereas if $\eta_z \le 5(M+1)/j_n$, then $e^{5\beta(M+1)}e^{-\beta j_n\eta_z} \ge 1$. Take $c_1 = 2\vee e^{5\beta(M+1)}$, for any $z\in B(0,3)$, we have

$$\max_{j_n-1\le k\le j_n+M-1} P[\mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(z,y_2,1)) \ge j_n-1 | \mathcal{H}_{nr_n^2(1-q_n)(1-v_n)}(S(0,y_1,1)) = k]$$

$$\le c_1e^{-\beta j_n\eta_z}.$$

Therefore the above discussion gives

$$h_n(z,y_1,y_2) \le \xi_n\Big(\frac{\mu_n}{j_n}\Big)^M + 2c_1\xi_n\frac{j_n}{\mu_n}e^{-\beta j_n \eta_z}$$

for n large enough. Now since $\inf_{z\in B(0,3)}\{\eta_z/||z||\} > 0$, there exists a constant $\gamma > 0$ such that

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi}\int_{B(0,3)} h_n(z,y_1,y_2) - \xi_n\Big(\frac{\mu_n}{j_n}\Big)^M \mathrm{d}z\mathrm{d}y_1\mathrm{d}y_2 &\le 8\pi^2 c_1\xi_n\big(\frac{j_n}{\mu_n}\big)\int_{B(0,3)} e^{-\gamma j_n ||z||}\mathrm{d}z \\ &\le c'\xi_n\big(\frac{j_n}{\mu_n}\big)\Gamma(2)/(\gamma j_n)^2.\end{aligned}$$

Accordingly, by the choice of M,

$$((1-v_n)n\xi_n)^{-1}I_{2,n}^{out} \le c((1-v_n)n\xi_n)^{-1}\cdot\mu_n n\xi_n\Big[\big(\frac{\mu_n}{j_n}\big)^M + \frac{1}{\mu_n j_n}\Big] \to 0,$$

as $n\to\infty$. The out-degree case hereby follows from (4).

For in-degree, set

$$W_n'^{in} := \sum_{i=1}^{N_{n(1-v_n)}} 1_{[\#\{X_j\in\mathcal{P}_{n(1-v_n)(1-q_n)}|X_i\in S(X_j,Y_j,r_n)\}\ge j_n+1]\cap[X_i\in Q]}$$

Then by Palm theory for Poisson process,

$$EW_n'^{in} \sim (1-v_n)n\int_Q P[Poi\big(\frac{\theta\alpha}{2\pi}nr_n^2(1-v_n)(1-q_n)\big)\ge j_n]\mathrm{d}x = (1-v_n)n\xi_n,$$

as $n\to\infty$. By Lemma 1, we thereby obtain

$$\big|P(W_n'^{in}=0) - e^{-EW_n'^{in}}\big| \le \min\Big(1,\frac{2}{(1-v_n)n\xi_n}\Big)(I_{1,n}^{in}+I_{2,n}^{in}), \tag{5}$$

for large enough n. We have

$$\begin{aligned}I_{1,n}^{in} &= (1-v_n)^2n^2\int_{\mathbb{R}^2} P[\tilde{\mathcal{P}}_{n(1-q_n)(1-v_n)}(B(x_1,r_n))\ge j_n] \\ &\quad \int_{B(x_1,3r_n)} P[\tilde{\mathcal{P}}_{n(1-q_n)(1-v_n)}(B(x_2,r_n))\ge j_n]f(x_2)\mathrm{d}x_2 f(x_1)\mathrm{d}x_1 \\ &\le n^2\xi_n^2\theta(3r_n)^2\end{aligned}$$

Thus, by (3), $((1-v_n)n\xi_n)^{-1}I_{1,n}^{in} \le c\mu_n\xi_n \to 0$. On the other hand, we have,

$$\begin{aligned}I_{2,n}^{in} &= (1-v_n)^2n^2\int_Q\int_{Q\cap B(x_1,3r_n)} P\big[\{\tilde{\mathcal{P}}_{n(1-q_n)(1-v_n)}(B(x_1,r_n)) \\ &\quad +1_{[x_1\in S(x_2,Y_2,r_n)]\cap[(x_2,x_1)\ not\ fails]}\ge j_n\}\cap\{\tilde{\mathcal{P}}_{n(1-q_n)(1-v_n)}(B(x_2,r_n)) \\ &\quad +1_{[x_2\in S(x_1,Y_1,r_n)]\cap[(x_1,x_2)\ not\ fails]}\ge j_n\}\big]\mathrm{d}x_2\mathrm{d}x_1 \\ &\le (1-v_n)^2n^2\int_Q\int_{Q\cap B(0,3r_n)} P\big[\{\tilde{\mathcal{P}}_{n(1-q_n)(1-v_n)}(B(0,r_n))\ge j_n-1\} \\ &\quad \cap\{\tilde{\mathcal{P}}_{n(1-q_n)(1-v_n)}(B(x_2-x_1,r_n))\ge j_n-1\}\big]\mathrm{d}x_2\mathrm{d}x_1 \\ &\le (1-v_n)^2n^2\int_{B(0,3)} g_n(z)\mathrm{d}z\end{aligned}$$

where,

$$g_n(z) := P\big[\{\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(0,1)) \geq j_n - 1\} \\ \cap \{\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(z,1)) \geq j_n - 1\}\big].$$

Also by (3), we choose M as above. Then we have

$$P[\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(0,1)) \geq j_n + M] \leq \xi_n\Big(\frac{\mu_n}{j_n}\Big)^M$$

and

$$P[\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(0,1)) \in \{j_n-1, j_n, \cdots, j_n+M-1\}] \leq 2\xi_n\frac{j_n}{\mu_n}$$

since $P[\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(0,1)) = j_n - 1] \leq \xi_n(j_n/\mu_n)$ and $P[\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(0,1)) \geq j_n] \leq \xi_n(j_n/\mu_n)$ when n is large enough. Let $\delta_z := |B(0,1)\backslash B(z,1)|/\theta$, hereby the conditional distribution of $\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(z,1))$, given that $\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(0,1)) = j_n + M$ is the sum of two independent random variables $j_n + M - U$ and V, where $U \sim Bin(j_n{+}M, \delta_z)$ and $V \sim Poi(\theta\alpha nr_n^2\delta_z(1{-}q_n)(1{-}v_n)/2\pi)$. Provided n is large enough so that $M + 1 < j_n\delta_z/5$, if $U > 3j_n\delta_z/5$ and $V < j_n\delta_z/5$ then $j_n + M - U + V < j_n - 1$. Now, note that $P[\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(z,1)) \geq j_n - 1|\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(0,1)) = k]$ is an increasing function of k, and by Chernoff bounds, there exists a constant $\beta > 0$, for any $z \in B(0,3)$ and n large enough, if $\delta_z > 5(M+1)/j_n$ then

$$\begin{aligned}\max_{j_n-1\leq k\leq j_n+M-1} & P[\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(z,1)) \geq j_n - 1|\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(0,1)) = k] \\ & \leq P[Bin(j_n + M, \delta_z) \leq 3j_n\delta_z/5] + P[Poi(\theta\alpha nr_n^2\delta_z(1-q_n)(1-v_n)/2\pi) \geq j_n\delta_z/5] \\ & \leq 2e^{-\beta j_n\delta_z},\end{aligned}$$

whereas if $\delta_z \leq 5(M+1)/j_n$, then $e^{5\beta(M+1)}e^{-\beta j_n\delta_z} \geq 1$. Take $c_2 = 2 \vee e^{5\beta(M+1)}$, for any $z \in B(0,3)$, we have

$$\max_{j_n-1\leq k\leq j_n+M-1} P[\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(z,1)) \geq j_n - 1|\mathcal{H}_{\frac{\alpha}{2\pi}nr_n^2(1-q_n)(1-v_n)}(B(0,1)) = k] \\ \leq c_2e^{-\beta j_n\delta_z}.$$

Consequently the above discussion gives

$$g_n(z) \leq \xi_n\Big(\frac{\mu_n}{j_n}\Big)^M + 2c_2\xi_n\frac{j_n}{\mu_n}e^{-\beta j_n\delta_z}$$

for n large enough. Now since $\inf_{z\in B(0,3)}\{\delta_z/||z||\} > 0$, there exists a constant $\gamma > 0$ such that

$$\begin{aligned}\int_{B(0,3)} g_n(z) - \xi_n\Big(\frac{\mu_n}{j_n}\Big)^M \mathrm{d}z & \leq 2c_2\xi_n\big(\frac{j_n}{\mu_n}\big)\int_{B(0,3)} e^{-\gamma j_n||z||}\mathrm{d}z \\ & \leq c'\xi_n\big(\frac{j_n}{\mu_n}\big)\Gamma(2)/(\gamma j_n)^2.\end{aligned}$$

Thus argue as the out-degree case, $((1-v_n)n\xi_n)^{-1}I_{2,n}^{in}$ tends to 0, as $n \to \infty$. We hereby complete the proof by using (5). □

Now we extend Proposition 1 from $\mathcal{P}_n$ to $\mathcal{X}_n$.

Proposition 2. *Let* $\mu_n := \frac{\alpha}{2} n r_n^2 (1-v_n)(1-q_n)$. *Suppose* $\inf_{n>0} \mu_n > 0$ *and* $\lim_{n\to\infty} \frac{\mu_n}{n^{1/6}} = 0$. *Suppose* $\{j_n\}_{n\geq 1}$ *is an* $\mathbb{N}-$*valued sequence such that for some* $\varepsilon > 0$, *(3) holds. Set* $\xi_n := P(Poi(\mu_n) \geq j_n)$. *Then*

$$\lim_{n\to\infty} [P(\Delta_n^{out} < j_n) - e^{-n(1-v_n)\xi_n}] = 0$$

and

$$\lim_{n\to\infty} [P(\Delta_n^{in} < j_n) - e^{-n(1-v_n)\xi_n}] = 0.$$

Proof. For out-degree, we first assume that $j_n \geq n^{1/5}$. We have

$$P(\Delta_n^{out} \geq n^{1/5}) \leq nP(Bin(n-1, \frac{\mu_n}{n}) \geq n^{1/5}) \to 0,$$

as $n \to \infty$, by Chernoff bounds. Accordingly, $P(\Delta_n^{out} \geq j_n)$ and $-n(1-v_n)\xi_n$ tend to zero. The result then follows.

From now on, we thereby assume $j_n < n^{1/5}$ for all n, without loss of generality. Set $\lambda_n := n + n^{3/4}$, and let $\mathcal{P}_{\lambda_n}$ be the Poisson point process coupled to $\mathcal{X}_n$ with intensity $\lambda_n f$. Denote by $\Delta_n^{+,out}$ the maximum out-degree in $G_\alpha(\mathcal{P}_{\lambda_n}, \mathcal{Y}_{N_{\lambda_n}}, v_n, q_n, r_n)$. Set $\mu_n^+ := \frac{\theta\alpha}{2\pi} \lambda_n r_n^2 (1-v_n)(1-q_n)$ and $\xi_n^+ := P(Poi(\mu_n^+) \geq j_n)$. Using Proposition 1 we have

$$\lim_{n\to\infty} [P(\Delta_n^{+,out} < j_n) - e^{-\lambda_n(1-v_n)\xi_n^+}] = 0. \tag{6}$$

Since $1 \leq (\mu_n^+/\mu_n)^{j_n} = (1+n^{-1/4})^{j_n} \to 1$ and $0 \leq \mu_n^+ - \mu_n = n^{-1/4}\mu_n \to 0$, as $n \to \infty$, we get

$$1 \leq \frac{\xi_n^+}{\xi_n} \leq \frac{e^{-\mu_n^+}(\mu_n^+)^{j_n}[1 + \frac{\mu_n^+}{j_n} + (\frac{\mu_n^+}{j_n})^2 + \cdots]}{e^{-\mu_n}\mu_n^{j_n}} \to 1.$$

Then by setting $a_n = n(1-v_n)\xi_n$ and $b_n = \frac{\lambda_n \xi_n^+}{n\xi_n} - 1$, we have $b_n > 0$ and $b_n \to 0$. Observe that $1 - e^{-a_n b_n} \leq 1 - e^{-\sqrt{b_n}} \to 0$, if $a_n \leq 1/\sqrt{b_n}$, while $e^{-a_n} < e^{-1/\sqrt{b_n}} \to 0$ if $a_n > 1/\sqrt{b_n}$. Consequently,

$$e^{-n(1-v_n)\xi_n} - e^{-\lambda_n(1-v_n)\xi_n^+} = e^{-a_n}(1 - e^{-a_n b_n}) \to 0,$$

as $n \to \infty$. Combing this with (6), we have

$$\lim_{n\to\infty} [P(\Delta_n^{+,out} < j_n) - e^{-n(1-v_n)\xi_n}] = 0.$$

Since $P(\Delta_n^{+,out} < j_n) - P(\Delta_n^{out} < j_n) = P(\Delta_n^{+,out} < j_n \leq \Delta_n^{out}) - P(\Delta_n^{out} < j_n \leq \Delta_n^{+,out} | n \leq N_{\lambda_n} \leq n + 2n^{3/4}) \cdot P(n \leq N_{\lambda_n} \leq n + 2n^{3/4})$, and $P(\Delta_n^{+,out} < j_n \leq \Delta_n^{out})$ tends to 0, $P(n \leq N_{\lambda_n} \leq n + 2n^{3/4})$ tends to 1 by Chebyshev inequality, as $n \to \infty$, to prove the result it suffices to prove that

$$\lim_{n\to\infty} P(\Delta_n^{out} < j_n \leq \Delta_n^{+,out} | n \leq N_{\lambda_n} \leq n + 2n^{3/4}) = 0.$$

Now suppose $\Delta_n^{+,out} \geq j_n$ and $n \leq N_{\lambda_n} \leq n + 2n^{3/4}$, then there exists a point in $\mathcal{P}_{\lambda_n}$ of out-degree at least j_n in $G_\alpha(\mathcal{P}_{\lambda_n}, \mathcal{Y}_{N_{\lambda_n}}, v_n, q_n, r_n)$. Therefore

$$P[j_n > \Delta_n^{out} | \Delta_n^{+,out} \geq j_n,\ n \leq N_{\lambda_n} \leq n + 2n^{3/4}] \leq (j_n + 1)\frac{2n^{3/4}}{n} \to 0.$$

The out-degree case thereby follows by multiplication formula of probability.

For in-degree, the same argument may be applied. □

Proof of Theorem 1. Let $\xi_n(j) := P(Poi(\mu_n) \geq j)$, then for $n \in \mathbb{N}$, take j_n satisfying $n\xi_n(j_n - 1) > (1 - v_n)^{-1} \geq n\xi_n(j_n)$. Set

$$k_n := \begin{cases} j_n - 1 & , if\ (1 - v_n)n\xi_n(j_n) \leq \sqrt{\frac{\xi_n(j_n)}{\xi_n(j_n - 1)}} \\ j_n & , otherwise \end{cases}$$

Take $\eta > 0$ satisfying $(1 + \varepsilon)^{-1} = 1 - 2\eta$. Let $i_n := \lfloor \mu_n(\ln n)^\eta \rfloor$, then $i_n/(\ln n)^{1-\eta} \to 0$, as $n \to \infty$. Hence, by Stirling formula,

$$(1 - v_n)n\xi_n(i_n) \geq (1 - v_n)ne^{-1/12i_n} \frac{1}{\sqrt{2\pi i_n}} e^{-i_n \ln(i_n/\mu_n)} \geq cni_n^{-1/2} e^{-i_n \ln(i_n/\mu_n)} \to \infty.$$

Thereby, $j_n \geq i_n$ and $j_n/\mu_n \to \infty$, as $n \to \infty$. Hence, $\xi_n(j_n)/\xi_n(j_n - 1)$, $\xi_n(j_n + 1)/\xi_n(j_n)$ and $\xi_n(j_n - 1)/\xi_n(j_n - 2)$ all tend to zero. By the definition of k_n, $(1 - v_n)n\xi_n(k_n + 1) \to 0$ and $(1 - v_n)n\xi_n(k_n - 1) \to \infty$, as $n \to \infty$. Consequently, by Proposition 1, we have

$$P(\Delta_n'^{out} < k_n + 1) \to 1, \quad P(\Delta_n'^{out} < k_n - 1) \to 0, \quad P(\Delta_n'^{out} < k_n) - e^{-(1-v_n)n\xi_n(k_n)} \to 0,$$

and

$$P(\Delta_n'^{in} < k_n + 1) \to 1, \quad P(\Delta_n'^{in} < k_n - 1) \to 0, \quad P(\Delta_n'^{in} < k_n) - e^{-(1-v_n)n\xi_n(k_n)} \to 0.$$

Also by Proposition 2, we have

$$P(\Delta_n^{out} < k_n + 1) \to 1, \quad P(\Delta_n^{out} < k_n - 1) \to 0, \quad P(\Delta_n^{out} < k_n) - e^{-(1-v_n)n\xi_n(k_n)} \to 0,$$

and

$$P(\Delta_n^{in} < k_n + 1) \to 1, \quad P(\Delta_n^{in} < k_n - 1) \to 0, \quad P(\Delta_n^{in} < k_n) - e^{-(1-v_n)n\xi_n(k_n)} \to 0.$$

The proof thus complete. □

4. Open problems

A natural question would be to ask what happens for other limiting regime of μ_n. We believe if $\ln n \ll \mu_n \ll (\ln n)^2$ and some regular conditions hold for faulty probabilities, then there exist sequences i_n, j_n such that for all x:

$$P\Big(\frac{\Delta_n^{out/in} - i_n}{j_n} < x\Big) \to e^{-e^{-x}}.$$

Therefore the focusing results will hold no longer. Results from extreme value theory suggest that might be the case. In Erdös-Rényi random graph, a similar result holds [4].

Of course it would be of interest to consider the density function other than the uniform one.

Note that our random faulty scaled sector graphs are still static models, so what can be said about the behavior of maximum degrees of a dynamic model? A direct and meaningful way to get a dynamic faulty scaled sector graph is to give every point X_i a random lifetime T_i. Suppose that these lifetimes are independent random variables with common distribution $F(t) = P(T_i \leq t)$.

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